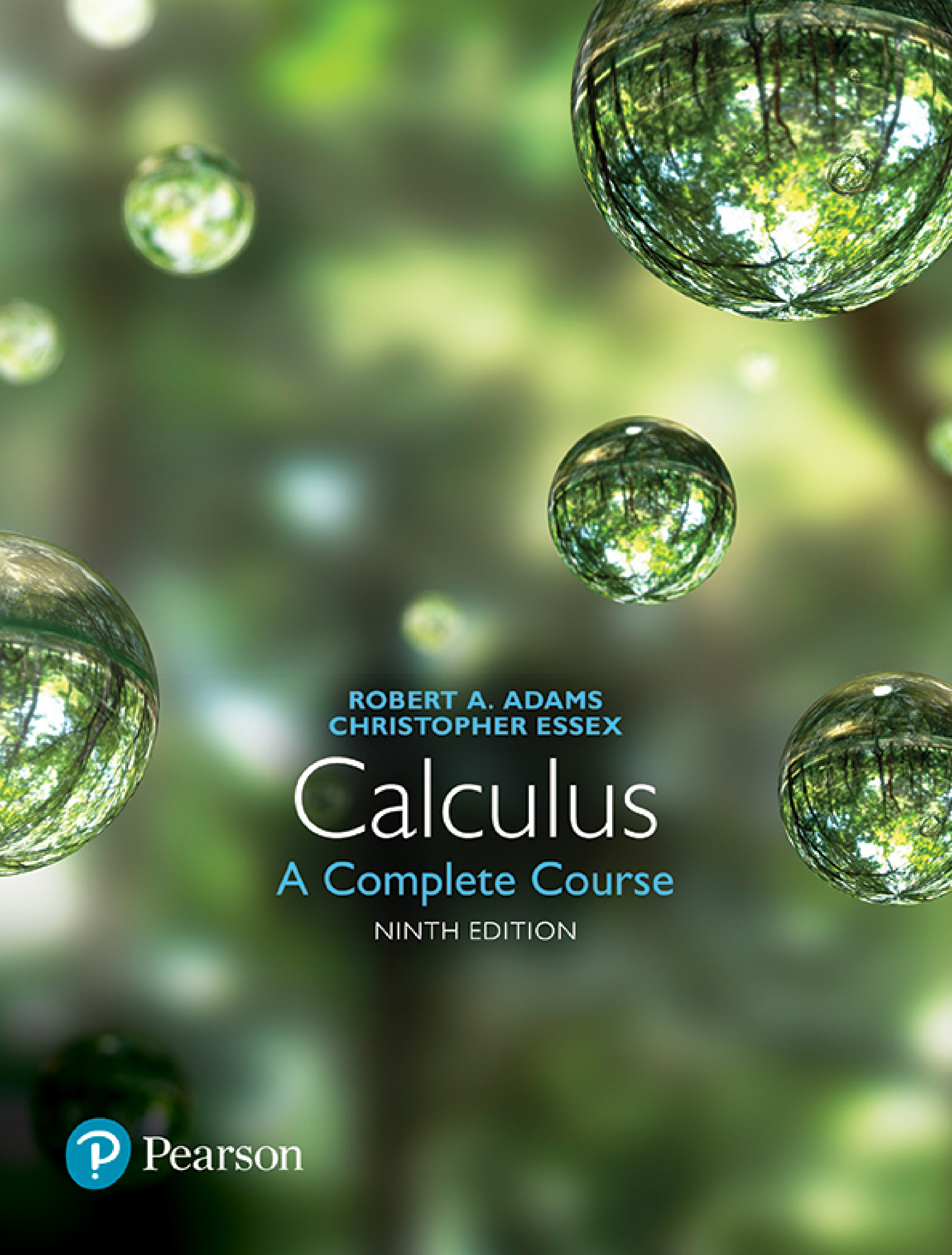

ROBERT A. ADAMS
CHRISTOPHER ESSEX

Calculus
A Complete Course

NINTH EDITION

Pearson

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ROBERT A. ADAMS
University of British Columbia

CHRISTOPHER ESSEX
University of Western Ontario

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A Complete Course

NINTH EDITION

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To Noreen and Sheran

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Preface

A fashionable curriculum proposition is that students should be given what they need and no more. It often comes bundled with language like "efficient" and "lean." Followers are quick to enumerate a number of topics they learned as students, which remained unused in their subsequent lives. What could they have accomplished, they muse, if they could have back the time lost studying such retrospectively unused topics? But many go further—they conflate unused with useless and then advocate that students should therefore have lean and efficient curricula, teaching only what students need. It has a convincing ring to it. Who wants to spend time on courses in "useless studies?"

When confronted with this compelling position, an even more compelling reply is to look the protagonist in the eye and ask, "How do you know what students need?" That's the trick, isn't it? If you could answer questions like that, you could become rich by making only those lean and efficient investments and bets that make money. It's more than that though. Knowledge of the fundamentals, unlike old lottery tickets, retains value. Few forms of human knowledge can beat mathematics in terms of enduring value and raw utility. Mathematics learned that you have not yet used retains value into an uncertain future.

It is thus ironic that the mathematics curriculum is one of the first topics that terms like *lean* and *efficient* get applied to. While there is much to discuss about this paradox, it is safe to say that it has little to do with what students actually need. If anything, people need more mathematics than ever as the arcane abstractions of yesteryear become the consumer products of today. Can one understand how web search engines work without knowing what an eigenvector is? Can one understand how banks try to keep your accounts safe on the web without understanding polynomials, or grasping how GPS works without understanding differentials?

All of this knowledge, seemingly remote from our everyday lives, is actually at the core of the modern world. Without mathematics you are estranged from it, and everything descends into rumour, superstition, and magic. The best lesson one can teach students about what to apply themselves to is that the future is uncertain, and it is a gamble how one chooses to spend one's efforts. But a sound grounding in mathematics is always a good first option. One of the most common educational regrets of many adults is that they did not spend enough time on mathematics in school, which is quite the opposite of the efficiency regrets of spending too much time on things unused.

A good mathematics textbook cannot be about a contrived minimal necessity. It has to be more than crib notes for a lean and diminished course in what students are deemed to need, only to be tossed away after the final exam. It must be more than a website or a blog. It should be something that stays with you, giving help in a familiar voice when you need to remember mathematics you will have forgotten over the years. Moreover, it should be something that one can grow into. People mature mathematically. As one does, concepts that seemed incomprehensible eventually become obvious. When that happens, new questions emerge that were previously inconceivable. This text has answers to many of those questions too.

Such a textbook must not only take into account the nature of the current audience, it must also be open to how well it bridges to other fields and introduces ideas new to the conventional curriculum. In this regard, this textbook is like no other. Topics not available in any other text are bravely introduced through the thematic concept of *gateway applications*. Applications of calculus have always been an important feature of earlier editions of this book. But the agenda of introducing gateway applications was introduced in the 8th edition. Rather than shrinking to what is merely needed, this 9th edition is still more comprehensive than the 8th edition. Of course, it remains possible to do a light and minimal treatment of the subject with this book, but the decision as to what that might mean precisely becomes the responsibility of a skilled instructor, and not the result of the limitations of some text. Correspondingly, a richer treatment is also an option. Flexibility in terms of emphasis, exercises, and projects is made easily possible with a larger span of subject material.

Some of the unique topics naturally addressed in the gateway applications, which may be added or omitted, include Liapunov functions, and Legendre transformations, not to mention exterior calculus. Exterior calculus is a powerful refinement of the calculus of a century ago, which is often overlooked. This text has a complete chapter on it, written accessibly in classical textbook style rather than as an advanced monograph. Other gateway applications are easy to cover in passing, but they are too often overlooked in terms of their importance to modern science. Liapunov functions are often squeezed into advanced books because they are left out of classical curricula, even though they are an easy addition to the discussion of vector fields, where their importance to stability theory and modern biomathematics can be usefully noted. Legendre transformations, which are so important to modern physics and thermodynamics, are a natural and easy topic to add to the discussion of differentials in more than one variable.

There are rich opportunities that this textbook captures. For example, it is the only mainstream textbook that covers sufficient conditions for maxima and minima in higher dimensions, providing answers to questions that most books gloss over. None of these are inaccessible. They are rich opportunities missed because many instructors are simply unfamiliar with their importance to other fields. The 9th edition continues in this tradition. For example, in the existing section on probability there is a new gateway application added that treats heavy-tailed distributions and their consequences for real-world applications.

The 9th edition, in addition to various corrections and refinements, fills in gaps in the treatment of differential equations from the 8th edition, with entirely new material. A linear operator approach to understanding differential equations is added. Also added is a refinement of the existing material on the Dirac delta function, and a full treatment of Laplace transforms. In addition, there is an entirely new section on phase plane analysis. The new phase plane section covers the classical treatment, if that is all one wants, but it goes much further for those who want more, now or later. It can set the reader up for dynamical systems in higher dimensions in a unique, lucid, and compact exposition. With existing treatments of various aspects of differential equations throughout the existing text, the 9th edition becomes suitable for a semester course in differential equations, in addition to the existing standard material suitable for four semesters of calculus.

Not only can the 9th edition be used to deliver five standard courses of conventional material, it can do much more through some of the unique topics and approaches mentioned above, which can be added or overlooked by the instructor without penalty. There is no other calculus book that deals better with computers and mathematics through Maple, in addition to unique but important applications from information theory to Lévy distributions, and does all of these things fearlessly. This 9th edition is the first one to be produced in full colour, and it continues to aspire to its subtitle: "A Complete Course." It is like no other.

About the Cover

The fall of rainwater droplets in a forest is frozen in an instant of time. For any small droplet of water, surface tension causes minimum energy to correspond to minimum surface area. Thus, small amounts of falling water are enveloped by nearly perfect minimal spheres, which act like lenses that image the forest background. The forest image is inverted because of the geometry of ray paths of light through a sphere. Close examination reveals that other droplets are also imaged, appearing almost like bubbles in glass. Still closer examination shows that the forest is right side up in the droplet images of the other droplets—transformation and inverse in one picture. If the droplets were much smaller, simple geometry of ray paths through a sphere would fail, because the wave nature of light would dominate. Interactions with the spherical droplets are then governed by Maxwell's equations instead of simple geometry. Tiny spheres exhibit Mie scattering of light instead, making a large collection of minute droplets, as in a cloud, seem brilliant white on a sunny day. The story of clouds, waves, rays, inverses, and minima are all contained in this instant of time in a forest.

To the Student

You are holding what has become known as a "high-end" calculus text in the book trade. You are lucky. Think of it as having a high-end touring car instead of a compact economy car. But, even though this is the first edition to be published in full colour, it is not high end in the material sense. It does not have scratch-and-sniff pages, sparkling radioactive ink, or anything else like that. It's the content that sets it apart. Unlike the car business, "high-end" book content is not priced any higher than that of any other book. It is one of the few consumer items where anyone can afford to buy into the high end. But there is a catch. Unlike cars, you have to do the work to achieve the promise of the book. So in that sense "high end" is more like a form of "secret" martial arts for your mind that the economy version cannot deliver. If you practise, your mind will become stronger. You will become more confident and disciplined. Secrets of the ages will become open to you. You will become fearless, as your mind longs to tackle any new mathematical challenge.

But hard work is the watchword. Practise, practise, practise. It is exhilarating when you finally get a new idea that you did not understand before. There are few experiences as great as figuring things out. Doing exercises and checking your answers against those in the back of the book are how you practise mathematics with a text. You can do essentially the same thing on a computer; you still do the problems and check the answers. However you do it, more exercises mean more practice and better performance.

There are numerous exercises in this text—too many for you to try them all perhaps, but be ambitious. Some are "drill" exercises to help you develop your skills in calculation. More important, however, are the problems that develop reasoning skills and your ability to apply the techniques you have learned to concrete situations. In some cases, you will have to plan your way through a problem that requires several different "steps" before you can get to the answer. Other exercises are designed to extend the theory developed in the text and therefore enhance your understanding of the concepts of calculus. Think of the problems as a tool to help you correctly wire your mind. You may have a lot of great components in your head, but if you don't wire the components together properly, your "home theatre" won't work.

The exercises vary greatly in difficulty. Usually, the more difficult ones occur toward the end of exercise sets, but these sets are not strictly graded in this way because exercises on a specific topic tend to be grouped together. Also, "difficulty" can be subjective. For some students, exercises designated difficult may seem easy, while exercises designated easy may seem difficult. Nonetheless, some exercises in the regular sets are marked with the symbols !, which indicates that the exercise is somewhat more difficult than most, or ?, which indicates a more theoretical exercise. The theoretical ones need not be difficult; sometimes they are quite easy. Most of the problems in the *Challenging Problems* section forming part of the *Chapter Review* at the end of most chapters are also on the difficult side.

It is not a bad idea to review the background material in Chapter P (Preliminaries), even if your instructor does not refer to it in class.

If you find some of the concepts in the book difficult to understand, *re-read* the material slowly, if necessary several times; *think about it*; formulate questions to ask fellow students, your TA, or your instructor. Don't delay. It is important to resolve your problems as soon as possible. If you don't understand today's topic, you may not understand how it applies to tomorrow's either. Mathematics builds from one idea to the next. Testing your understanding of the later topics also tests your understanding of the earlier ones. Do not be discouraged if you can't do *all* the exercises. Some are very difficult indeed. The range of exercises ensures that nearly all students can find a comfortable level to practise at, while allowing for greater challenges as skill grows.

Answers for most of the odd-numbered exercises are provided at the back of the book. Exceptions are exercises that don't have short answers: for example, "Prove that ..." or "Show that ..." problems where the answer is the whole solution. A *Student Solutions Manual* that contains detailed solutions to even-numbered exercises is available.

Besides ! and ? used to mark more difficult and theoretical problems, the following symbols are used to mark exercises of special types:

- Exercises pertaining to differential equations and initial-value problems. (It is not used in sections that are wholly concerned with DEs.)
- Problems requiring the use of a calculator. Often a scientific calculator is needed. Some such problems may require a programmable calculator.
- Problems requiring the use of either a graphing calculator or mathematical graphing software on a personal computer.
- Problems requiring the use of a computer. Typically, these will require either computer algebra software (e.g., Maple, Mathematica) or a spreadsheet program such as Microsoft Excel.

To the Instructor

Calculus: a Complete Course, 9th Edition contains 19 chapters, P and 1–18, plus 5 Appendices. It covers the material usually encountered in a three- to five-semester real-variable calculus program, involving real-valued functions of a single real variable (differential calculus in Chapters 1–4 and integral calculus in Chapters 5–8), as well as vector-valued functions of a single real variable (covered in Chapter 11), real-valued functions of several real variables (in Chapters 12–14), and vector-valued functions of several real variables (in Chapters 15–17). Chapter 9 concerns sequences and series, and its position is rather arbitrary.

Most of the material requires only a reasonable background in high school algebra and analytic geometry. (See Chapter P—Preliminaries for a review of this material.) However, some optional material is more subtle and/or theoretical and is intended for stronger students, special topics, and reference purposes. It also allows instructors considerable flexibility in making points, answering questions, and selective enrichment of a course.

Chapter 10 contains necessary background on vectors and geometry in 3-dimensional space as well as some linear algebra that is useful, although not absolutely essential, for the understanding of subsequent multivariable material. Material on differential equations is scattered throughout the book, but Chapter 18 provides a compact treatment of ordinary differential equations (ODEs), which may provide enough material for a one-semester course on the subject.

There are two split versions of the complete book. *Single-Variable Calculus, 9th Edition* covers Chapters P, 1–9, 18 and all five appendices. *Calculus of Several Variables, 9th Edition* covers Chapters 9–18 and all five appendices. It also begins with a brief review of Single-Variable Calculus.

Besides numerous improvements and clarifications throughout the book and tweakings of existing material such as consideration of probability densities with heavy tails in Section 7.8, and a less restrictive definition of the Dirac delta function in Section 16.1, there are two new sections in Chapter 18, one on Laplace Transforms (Section 18.7) and one on Phase Plane Analysis of Dynamical Systems (Section 18.9).

There is a wealth of material here—too much to include in any one course. It was never intended to be otherwise. You must select what material to include and what to omit, taking into account the background and needs of your students. At the University of British Columbia, where one author taught for 34 years, and at the University of Western Ontario, where the other author continues to teach, calculus is divided into four semesters, the first two covering single-variable calculus, the third covering functions of several variables, and the fourth covering vector calculus. In none of these courses was there enough time to cover all the material in the appropriate chapters; some sections are always omitted. The text is designed to allow students and instructors to conveniently find their own level while enhancing any course from general calculus to courses focused on science and engineering students.

Several supplements are available for use with *Calculus: A Complete Course, 9th Edition*. Available to students is the **Student Solutions Manual** (ISBN: 9780134491073): This manual contains detailed solutions to all the even-numbered exercises, prepared by the authors. There are also such Manuals for the split volumes, for *Single Variable Calculus* (ISBN: 9780134579863), and for *Calculus of Several Variables* (ISBN: 9780134579856).

Available to instructors are the following resources:

- **Instructor's Solutions Manual**
- **Computerized Test Bank** Pearson's computerized test bank allows instructors to filter and select questions to create quizzes, tests, or homework (over 1,500 test questions)
- **Image Library**, which contains all of the figures in the text provided as individual enlarged .pdf files suitable for printing to transparencies.

These supplements are available for download from a password-protected section of Pearson Canada's online catalogue (catalogue.pearsoned.ca). Navigate to this book's catalogue page to view a list of those supplements that are available. Speak to your local Pearson sales representative for details and access.

Also available to qualified instructors are **MyMathLab**® and **MathXL**® Online Courses for which access codes are required.

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The expunging of errors and obscurities in a text is an ongoing and asymptotic process; hopefully each edition is better than the previous one. Nevertheless, some such imperfections always remain, and we will be grateful to any readers who call them to our attention, or give us other suggestions for future improvements.

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R.A.A.
Vancouver, Canada
`adms@math.ubc.ca`

C.E.
London, Canada
`essex@uwo.ca`

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What Is Calculus?

Early in the seventeenth century, the German mathematician Johannes Kepler analyzed a vast number of astronomical observations made by Danish astronomer Tycho Brahe and concluded that the planets must move around the sun in elliptical orbits. He didn't know why. Fifty years later, the English mathematician and physicist Isaac Newton answered that question.

Why do the planets move in elliptical orbits around the sun? Why do hurricane winds spiral counterclockwise in the northern hemisphere? How can one predict the effects of interest rate changes on economies and stock markets? When will radioactive material be sufficiently decayed to enable safe handling? How do warm ocean currents in the equatorial Pacific affect the climate of eastern North America? How long will the concentration of a drug in the bloodstream remain at effective levels? How do radio waves propagate through space? Why does an epidemic spread faster and faster and then slow down? How can I be sure the bridge I designed won't be destroyed in a windstorm?

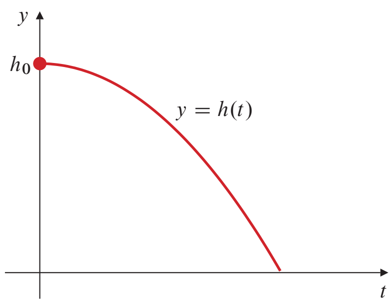
These and many other questions of interest and importance in our world relate directly to our ability to analyze motion and how quantities change with respect to time or each other. Algebra and geometry are useful tools for describing relationships between *static* quantities, but they do not involve concepts appropriate for describing how a quantity *changes*. For this we need new mathematical operations that go beyond the algebraic operations of addition, subtraction, multiplication, division, and the taking of powers and roots. We require operations that measure the way related quantities change.

Calculus provides the tools for describing motion quantitatively. It introduces two new operations called *differentiation* and *integration*, which, like addition and subtraction, are opposites of one another; what differentiation does, integration undoes.

For example, consider the motion of a falling rock. The height (in metres) of the rock t seconds after it is dropped from a height of h_0 m is a function $h(t)$ given by

$$h(t) = h_0 - 4.9t^2.$$

The graph of $y = h(t)$ is shown in the figure below:



The process of differentiation enables us to find a new function, which we denote $h'(t)$ and call the *derivative* of h with respect to t, which represents the *rate of change* of the height of the rock, that is, its *velocity* in metres/second:

$$h'(t) = -9.8t.$$

Conversely, if we know the velocity of the falling rock as a function of time, integration enables us to find the height function $h(t)$.

Calculus was invented independently and in somewhat different ways by two seventeenth-century mathematicians: Isaac Newton and Gottfried Wilhelm Leibniz. Newton's motivation was a desire to analyze the motion of moving objects. Using his calculus, he was able to formulate his laws of motion and gravitation and *conclude from them* that the planets must move around the sun in elliptical orbits.

Many of the most fundamental and important "laws of nature" are conveniently expressed as equations involving rates of change of quantities. Such equations are called *differential equations*, and techniques for their study and solution are at the heart of calculus. In the falling rock example, the appropriate law is **Newton's Second Law of Motion:**

$$\text{force} = \text{mass} \times \text{acceleration}.$$

The *acceleration*, -9.8 m/s^2, is the rate of change (the *derivative*) of the velocity, which is in turn the rate of change (the *derivative*) of the height function.

Much of mathematics is related indirectly to the study of motion. We regard *lines*, or *curves*, as geometric objects, but the ancient Greeks thought of them as paths traced out by moving points. Nevertheless, the study of curves also involves geometric concepts such as tangency and area. The process of differentiation is closely tied to the geometric problem of finding tangent lines; similarly, integration is related to the geometric problem of finding areas of regions with curved boundaries.

Both differentiation and integration are defined in terms of a new mathematical operation called a **limit**. The concept of the limit of a function will be developed in Chapter 1. That will be the real beginning of our study of calculus. In the chapter called "Preliminaries" we will review some of the background from algebra and geometry needed for the development of calculus.

CHAPTER P

Preliminaries

> 'Reeling and Writhing, of course, to begin with,' the Mock Turtle replied, 'and the different branches of Arithmetic—Ambition, Distraction, Uglification, and Derision.'

Lewis Carroll (Charles Lutwidge Dodgson) 1832–1898
from *Alice's Adventures in Wonderland*

Introduction

This preliminary chapter reviews the most important things you should know before beginning calculus. Topics include the real number system; Cartesian coordinates in the plane; equations representing straight lines, circles, and parabolas; functions and their graphs; and, in particular, polynomials and trigonometric functions.

Depending on your precalculus background, you may or may not be familiar with these topics. If you are, you may want to skim over this material to refresh your understanding of the terms used; if not, you should study this chapter in detail.

P.1 Real Numbers and the Real Line

Calculus depends on properties of the real number system. **Real numbers** are numbers that can be expressed as decimals, for example,

$$\begin{aligned} 5 &= 5.00000\ldots \\ -\tfrac{3}{4} &= -0.750000\ldots \\ \tfrac{1}{3} &= 0.3333\ldots \\ \sqrt{2} &= 1.4142\ldots \\ \pi &= 3.14159\ldots \end{aligned}$$

In each case the three dots (...) indicate that the sequence of decimal digits goes on forever. For the first three numbers above, the patterns of the digits are obvious; we know what all the subsequent digits are. For $\sqrt{2}$ and π there are no obvious patterns.

The real numbers can be represented geometrically as points on a number line, which we call the **real line**, shown in Figure P.1. The symbol $\mathbb{R}$ is used to denote either the real number system or, equivalently, the real line.

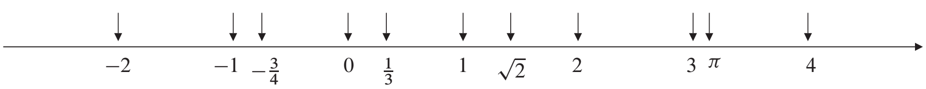


Figure P.1 The real line

The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. You are already familiar with the *algebraic properties*; roughly speaking, they assert that real numbers can be added, subtracted, multiplied, and divided (except by zero) to produce more real numbers and that the usual rules of arithmetic are valid.

The *order properties* of the real numbers refer to the order in which the numbers appear on the real line. If x lies to the left of y, then we say that "x is less than y" or "y is greater than x." These statements are written symbolically as $x < y$ and $y > x$, respectively. The inequality $x \leq y$ means that either $x < y$ or $x = y$. The order properties of the real numbers are summarized in the following *rules for inequalities*:

The symbol $\Longrightarrow$ means "implies."

Rules for inequalities

If a, b, and c are real numbers, then:

1. $a < b \quad \Longrightarrow \quad a + c < b + c$
2. $a < b \quad \Longrightarrow \quad a - c < b - c$
3. $a < b$ and $c > 0 \quad \Longrightarrow \quad ac < bc$
4. $a < b$ and $c < 0 \quad \Longrightarrow \quad ac > bc$; in particular, $-a > -b$
5. $a > 0 \quad \Longrightarrow \quad \dfrac{1}{a} > 0$
6. $0 < a < b \quad \Longrightarrow \quad \dfrac{1}{b} < \dfrac{1}{a}$

Rules 1–4 and 6 (for $a > 0$) also hold if $<$ and $>$ are replaced by $\leq$ and $\geq$.

Note especially the rules for multiplying (or dividing) an inequality by a number. If the number is positive, the inequality is preserved; if the number is negative, the inequality is reversed.

The *completeness* property of the real number system is more subtle and difficult to understand. One way to state it is as follows: if A is any set of real numbers having at least one number in it, and if there exists a real number y with the property that $x \leq y$ for every x in A (such a number y is called an **upper bound** for A), then there exists a *smallest* such number, called the **least upper bound** or **supremum** of A, and denoted $\sup(A)$. Roughly speaking, this says that there can be no holes or gaps on the real line—every point corresponds to a real number. We will not need to deal much with completeness in our study of calculus. It is typically used to prove certain important results—in particular, Theorems 8 and 9 in Chapter 1. (These proofs are given in Appendix III but are not usually included in elementary calculus courses; they are studied in more advanced courses in mathematical analysis.) However, when we study infinite sequences and series in Chapter 9, we will make direct use of completeness.

The set of real numbers has some important special subsets:

(i) the **natural numbers** or **positive integers**, namely, the numbers $1, 2, 3, 4, \ldots$

(ii) the **integers**, namely, the numbers $0, \pm 1, \pm 2, \pm 3, \ldots$

(iii) the **rational numbers**, that is, numbers that can be expressed in the form of a fraction m/n, where m and n are integers, and $n \neq 0$.

The rational numbers are precisely those real numbers with decimal expansions that are either:

(a) terminating, that is, ending with an infinite string of zeros, for example, $3/4 = 0.750000\ldots$, or

(b) repeating, that is, ending with a string of digits that repeats over and over, for example, $23/11 = 2.090909\ldots = 2.\overline{09}$. (The bar indicates the pattern of repeating digits.)

Real numbers that are not rational are called *irrational numbers*.

EXAMPLE 1 Show that each of the numbers (a) $1.323232\cdots = 1.\overline{32}$ and (b) $0.3405405405\ldots = 0.3\overline{405}$ is a rational number by expressing it as a quotient of two integers.

Solution

(a) Let $x = 1.323232\ldots$ Then $x - 1 = 0.323232\ldots$ and

$$100x = 132.323232\ldots = 132 + 0.323232\ldots = 132 + x - 1.$$

Therefore, $99x = 131$ and $x = 131/99$.

(b) Let $y = 0.3405405405\ldots$ Then $10y = 3.405405405\ldots$ and $10y - 3 = 0.405405405\ldots$ Also,

$$10,000y = 3,405.405405405\ldots = 3,405 + 10y - 3.$$

Therefore, $9,990y = 3,402$ and $y = 3,402/9,990 = 63/185$.

The set of rational numbers possesses all the algebraic and order properties of the real numbers but not the completeness property. There is, for example, no rational number whose square is 2. Hence, there is a "hole" on the "rational line" where $\sqrt{2}$ should be.[1] Because the real line has no such "holes," it is the appropriate setting for studying limits and therefore calculus.

Intervals

A subset of the real line is called an **interval** if it contains at least two numbers and also contains all real numbers between any two of its elements. For example, the set of real numbers x such that $x > 6$ is an interval, but the set of real numbers y such that $y \neq 0$ is not an interval. (Why?) It consists of two intervals.

If a and b are real numbers and $a < b$, we often refer to

(i) the **open interval** from a to b, denoted by (a, b), consisting of all real numbers x satisfying $a < x < b$.

(ii) the **closed interval** from a to b, denoted by $[a, b]$, consisting of all real numbers x satisfying $a \le x \le b$.

(iii) the **half-open interval** $[a, b)$, consisting of all real numbers x satisfying the inequalities $a \le x < b$.

(iv) the **half-open interval** $(a, b]$, consisting of all real numbers x satisfying the inequalities $a < x \le b$.

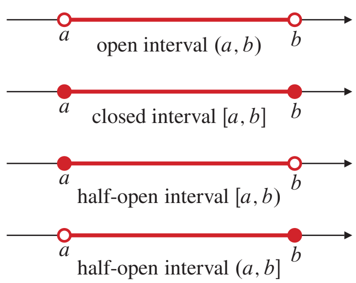


Figure P.2 Finite intervals

These are illustrated in Figure P.2. Note the use of hollow dots to indicate endpoints of intervals that are not included in the intervals, and solid dots to indicate endpoints that are included. The endpoints of an interval are also called **boundary points**.

The intervals in Figure P.2 are **finite intervals**; each of them has finite length $b - a$. Intervals can also have infinite length, in which case they are called **infinite intervals**. Figure P.3 shows some examples of infinite intervals. Note that the whole real line $\mathbb{R}$ is an interval, denoted by $(-\infty, \infty)$. The symbol ∞ ("infinity") does *not* denote a real number, so we never allow ∞ to belong to an interval.

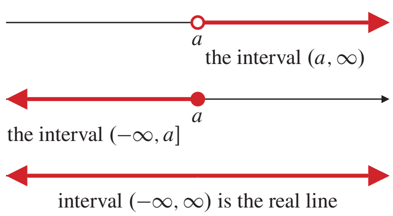


Figure P.3 Infinite intervals

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1 How do we know that $\sqrt{2}$ is an irrational number? Suppose, to the contrary, that $\sqrt{2}$ is rational. Then $\sqrt{2} = m/n$, where m and n are integers and $n \neq 0$. We can assume that the fraction m/n has been "reduced to lowest terms"; any common factors have been cancelled out. Now $m^2/n^2 = 2$, so $m^2 = 2n^2$, which is an even integer. Hence, m must also be even. (The square of an odd integer is always odd.) Since m is even, we can write $m = 2k$, where k is an integer. Thus $4k^2 = 2n^2$ and $n^2 = 2k^2$, which is even. Thus n is also even. This contradicts the assumption that $\sqrt{2}$ could be written as a fraction m/n in lowest terms; m and n cannot both be even. Accordingly, there can be no rational number whose square is 2.

EXAMPLE 2 Solve the following inequalities. Express the solution sets in terms of intervals and graph them.

(a) $2x - 1 > x + 3$ (b) $-\dfrac{x}{3} \ge 2x - 1$ (c) $\dfrac{2}{x-1} \ge 5$

Solution

(a)
$$\begin{aligned} 2x - 1 &> x + 3 && \text{Add 1 to both sides.} \\ 2x &> x + 4 && \text{Subtract } x \text{ from both sides.} \\ x &> 4 && \text{The solution set is the interval } (4, \infty). \end{aligned}$$

(b)
$$\begin{aligned} -\frac{x}{3} &\ge 2x - 1 && \text{Multiply both sides by } -3. \\ x &\le -6x + 3 && \text{Add } 6x \text{ to both sides.} \\ 7x &\le 3 && \text{Divide both sides by 7.} \\ x &\le \frac{3}{7} && \text{The solution set is the interval } (-\infty, 3/7]. \end{aligned}$$

(c) We transpose the 5 to the left side and simplify to rewrite the given inequality in an equivalent form:

$$\frac{2}{x-1} - 5 \ge 0 \iff \frac{2 - 5(x-1)}{x-1} \ge 0 \iff \frac{7-5x}{x-1} \ge 0.$$

The symbol $\iff$ means "if and only if" or "is equivalent to." If A and B are two statements, then $A \iff B$ means that the truth of either statement implies the truth of the other, so either both must be true or both must be false.

The fraction $\dfrac{7-5x}{x-1}$ is undefined at $x = 1$ and is 0 at $x = 7/5$. Between these numbers it is positive if the numerator and denominator have the same sign, and negative if they have opposite sign. It is easiest to organize this sign information in a chart:

x		1		7/5	
$7 - 5x$	$+$	$+$	$+$	0	$-$
$x - 1$	$-$	0	$+$	$+$	$+$
$(7 - 5x)/(x - 1)$	$-$	undef	$+$	0	$-$

Thus the solution set of the given inequality is the interval $(1, 7/5]$.

See Figure P.4 for graphs of the solutions.

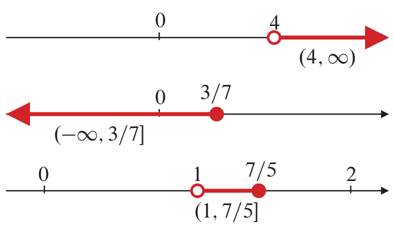

Figure P.4 The intervals for Example 2

Sometimes we will need to solve systems of two or more inequalities that must be satisfied simultaneously. We still solve the inequalities individually and look for numbers in the intersection of the solution sets.

EXAMPLE 3 Solve the systems of inequalities:
(a) $3 \le 2x + 1 \le 5$ (b) $3x - 1 < 5x + 3 \le 2x + 15$.

Solution

(a) Using the technique of Example 2, we can solve the inequality $3 \le 2x + 1$ to get $2 \le 2x$, so $x \ge 1$. Similarly, the inequality $2x + 1 \le 5$ leads to $2x \le 4$, so $x \le 2$. The solution set of system (a) is therefore the closed interval $[1, 2]$.

(b) We solve both inequalities as follows:

$$\left.\begin{aligned} 3x - 1 &< 5x + 3 \\ -1 - 3 &< 5x - 3x \\ -4 &< 2x \\ -2 &< x \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} 5x + 3 &\le 2x + 15 \\ 5x - 2x &\le 15 - 3 \\ 3x &\le 12 \\ x &\le 4 \end{aligned}\right.$$

The solution set is the interval $(-2, 4]$.

Solving quadratic inequalities depends on solving the corresponding quadratic equations.

EXAMPLE 4

Quadratic inequalities

Solve: (a) $x^2 - 5x + 6 < 0$ (b) $2x^2 + 1 > 4x$.

Solution

(a) The trinomial $x^2 - 5x + 6$ factors into the product $(x-2)(x-3)$, which is negative if and only if exactly one of the factors is negative. Since $x - 3 < x - 2$, this happens when $x - 3 < 0$ and $x - 2 > 0$. Thus we need $x < 3$ and $x > 2$; the solution set is the open interval $(2, 3)$.

(b) The inequality $2x^2+1 > 4x$ is equivalent to $2x^2-4x+1 > 0$. The corresponding quadratic equation $2x^2 - 4x + 1 = 0$, which is of the form $Ax^2 + Bx + C = 0$, can be solved by the quadratic formula (see Section P.6):

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{4 \pm \sqrt{16-8}}{4} = 1 \pm \frac{\sqrt{2}}{2},$$

so the given inequality can be expressed in the form

$$\left(x - 1 + \tfrac{1}{2}\sqrt{2}\right)\left(x - 1 - \tfrac{1}{2}\sqrt{2}\right) > 0.$$

This is satisfied if both factors on the left side are positive or if both are negative. Therefore, we require that either $x < 1 - \frac{1}{2}\sqrt{2}$ or $x > 1 + \frac{1}{2}\sqrt{2}$. The solution set is the *union* of intervals $\left(-\infty, 1 - \frac{1}{2}\sqrt{2}\right) \cup \left(1 + \frac{1}{2}\sqrt{2}, \infty\right)$.

Note the use of the symbol $\cup$ to denote the **union** of intervals. A real number is in the union of intervals if it is in at least one of the intervals. We will also need to consider the **intersection** of intervals from time to time. A real number belongs to the intersection of intervals if it belongs to *every one* of the intervals. We will use $\cap$ to denote intersection. For example,

$$[1, 3) \cap [2, 4] = [2, 3) \quad \text{while} \quad [1, 3) \cup [2, 4] = [1, 4].$$

EXAMPLE 5

Solve the inequality $\dfrac{3}{x-1} < -\dfrac{2}{x}$ and graph the solution set.

Solution We would like to multiply by $x(x-1)$ to clear the inequality of fractions, but this would require considering three cases separately. (What are they?) Instead, we will transpose and combine the two fractions into a single one:

$$\frac{3}{x-1} < -\frac{2}{x} \iff \frac{3}{x-1} + \frac{2}{x} < 0 \iff \frac{5x-2}{x(x-1)} < 0.$$

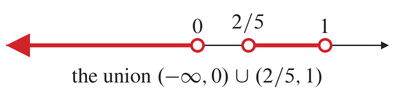

Figure P.5 The solution set for Example 5

We examine the signs of the three factors in the left fraction to determine where that fraction is negative:

x		0		$2/5$		1	
$5x-2$	$-$	$-$	$-$	0	$+$	$+$	$+$
x	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$\dfrac{5x-2}{x(x-1)}$	$-$	undef	$+$	0	$-$	undef	$+$

The solution set of the given inequality is the union of these two intervals, namely, $(-\infty, 0) \cup (2/5, 1)$. See Figure P.5.

The Absolute Value

The **absolute value**, or **magnitude**, of a number x, denoted $|x|$ (read "the absolute value of x"), is defined by the formula

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

The vertical lines in the symbol $|x|$ are called **absolute value bars**.

EXAMPLE 6 $|3| = 3, \quad |0| = 0, \quad |-5| = 5.$

Note that $|x| \ge 0$ for every real number x, and $|x| = 0$ only if $x = 0$. People sometimes find it confusing to say that $|x| = -x$ when x is negative, but this is correct since $-x$ is positive in that case. The symbol $\sqrt{a}$ always denotes the *nonnegative* square root of a, so an alternative definition of $|x|$ is $|x| = \sqrt{x^2}$.

It is important to remember that $\sqrt{a^2} = |a|$. Do not write $\sqrt{a^2} = a$ unless you already know that $a \ge 0$.

Geometrically, $|x|$ represents the (nonnegative) distance from x to 0 on the real line. More generally, $|x - y|$ represents the (nonnegative) distance between the points x and y on the real line, since this distance is the same as that from the point $x - y$ to 0 (see Figure P.6):

$$|x - y| = \begin{cases} x - y, & \text{if } x \ge y \\ y - x, & \text{if } x < y. \end{cases}$$

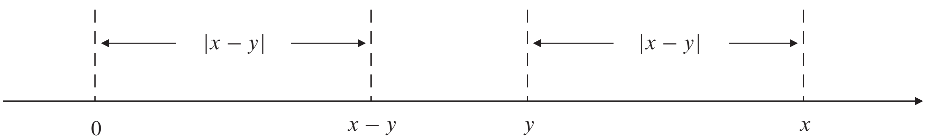

Figure P.6
$|x - y|$ = distance from x to y

The absolute value function has the following properties:

Properties of absolute values

1. $|-a| = |a|$. A number and its negative have the same absolute value.
2. $|ab| = |a||b|$ and $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$. The absolute value of a product (or quotient) of two numbers is the product (or quotient) of their absolute values.
3. $|a \pm b| \le |a| + |b|$ (the **triangle inequality**). The absolute value of a sum of or difference between numbers is less than or equal to the sum of their absolute values.

The first two of these properties can be checked by considering the cases where either of a or b is either positive or negative. The third property follows from the first two because $\pm 2ab \le |2ab| = 2|a||b|$. Therefore, we have

$$\begin{aligned} |a \pm b|^2 = (a \pm b)^2 &= a^2 \pm 2ab + b^2 \\ &\le |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2, \end{aligned}$$

and taking the (positive) square roots of both sides, we obtain $|a \pm b| \le |a| + |b|$. This result is called the "triangle inequality" because it follows from the geometric fact that the length of any side of a triangle cannot exceed the sum of the lengths of the other two sides. For instance, if we regard the points 0, a, and b on the number line as the vertices of a degenerate "triangle," then the sides of the triangle have lengths $|a|$, $|b|$, and $|a - b|$. The triangle is degenerate since all three of its vertices lie on a straight line.

Equations and Inequalities Involving Absolute Values

The equation $|x| = D$ (where $D > 0$) has two solutions, $x = D$ and $x = -D$: the two points on the real line that lie at distance D from the origin. Equations and inequalities involving absolute values can be solved algebraically by breaking them into cases according to the definition of absolute value, but often they can also be solved geometrically by interpreting absolute values as distances. For example, the inequality $|x - a| < D$ says that the distance from x to a is less than D, so x must lie between $a - D$ and $a + D$. (Or, equivalently, a must lie between $x - D$ and $x + D$.) If D is a positive number, then

$$\begin{aligned}
|x| = D &\iff \text{either } x = -D \text{ or } x = D\\
|x| < D &\iff -D < x < D\\
|x| \le D &\iff -D \le x \le D\\
|x| > D &\iff \text{either } x < -D \text{ or } x > D
\end{aligned}$$

More generally,

$$\begin{aligned}
|x - a| = D &\iff \text{either } x = a - D \text{ or } x = a + D\\
|x - a| < D &\iff a - D < x < a + D\\
|x - a| \le D &\iff a - D \le x \le a + D\\
|x - a| > D &\iff \text{either } x < a - D \text{ or } x > a + D
\end{aligned}$$

EXAMPLE 7 Solve: (a) $|2x + 5| = 3$ (b) $|3x - 2| \le 1$.

Solution

(a) $|2x + 5| = 3 \iff 2x + 5 = \pm 3$. Thus, either $2x = -3 - 5 = -8$ or $2x = 3 - 5 = -2$. The solutions are $x = -4$ and $x = -1$.

(b) $|3x - 2| \le 1 \iff -1 \le 3x - 2 \le 1$. We solve this pair of inequalities:

$$\left\{\begin{aligned} -1 &\le 3x - 2\\ -1 + 2 &\le 3x\\ 1/3 &\le x \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} 3x - 2 &\le 1\\ 3x &\le 1 + 2\\ x &\le 1 \end{aligned}\right\}.$$

Thus the solutions lie in the interval $[1/3, 1]$.

Remark Here is how part (b) of Example 7 could have been solved geometrically, by interpreting the absolute value as a distance:

$$|3x - 2| = \left|3\left(x - \frac{2}{3}\right)\right| = 3\left|x - \frac{2}{3}\right|.$$

Thus, the given inequality says that

$$3\left|x - \frac{2}{3}\right| \le 1 \qquad \text{or} \qquad \left|x - \frac{2}{3}\right| \le \frac{1}{3}.$$

This says that the distance from x to $2/3$ does not exceed $1/3$. The solutions for x therefore lie between $1/3$ and 1, including both of these endpoints. (See Figure P.7.)

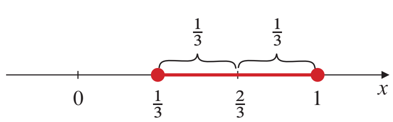

Figure P.7 The solution set for Example 7(b)